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Topological and metric properties of distributions of random variables represented by the alternating Lüroth series with independent elements

Mykola Pratsiovytyi and Yuriy Khvorostina

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Abstract. In the paper we consider the distributions of random variables represented by the alternating Lüroth series ($\widetilde{L}$-expansion). We study Lebesgue structure, topological, metric and fractal properties of these random variables. We prove that random variable with independent $\widetilde{L}$-symbols has a pure discrete, pure absolutely continuous or pure singularly continuous distribution. We describe topological and metric properties of the spectra of distributions of random variables as well as properties of their probability distribution functions.

Keywords. Expansions of numbers by alternating Lüroth series, geometry of $\widetilde{L}$-representation, absolutely continuous probability distribution, singular probability distribution, Lebesgue structure of probability distribution.

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Introduction

In 1883 J. Lüroth introduced the sign positive series expansion [4], members of which are inverse numbers to positive integers. J. Galambos, K. Dajani, C. Kraaikamp, C. Ganatsiou and others investigated the Lüroth sign positive expansion. S. Kalpazidou, A. Knopfmacher, and J. Knopfmacher introduced a Lüroth-type alternating expansion [2, 3]. They proved that for any real number $x \in (0, 1]$ there exists either finite tuple of positive integers $(a_1, a_2, \ldots, a_n)$ or a sequence of positive integers (a_n), $a_n = a_n(x)$, such that

$$x = \frac{1}{a_1} + \sum_{n \geq 2} \frac{(-1)^{n-1}}{a_1(a_1+1)\ldots a_{n-1}(a_{n-1}+1)a_n}. \tag{0.1}$$

Moreover, each irrational number has a unique infinite and non-periodic representation and each rational number has either finite or periodic representation.

Equality (0.1) is called the alternating Lüroth series representation or $\widetilde{L}$-expansion for number x. We will write symbolically

$$x = \Delta^{\widetilde{L}}_{a_1 a_2 \ldots a_k \ldots}.$$

Recently the present authors [6] introduced and studied the set of incomplete sums of the alternating Lüroth series and probability on it.

Let (η_k) be a sequence of independent random variables taking the values $1, 2, \ldots, i, \ldots$ with probabilities $p_{1k}, p_{2k}, \ldots, p_{ik}, \ldots$ respectively,

$$p_{ik} \geq 0, \quad p_{1k} + p_{2k} + \cdots = 1 \quad \text{for all } k \in \mathbb{N}.$$

This paper is devoted to study of the Lebesgue structure (content of discrete, absolutely continuous and singular components), topological, metric and fractal properties of the random variable

$$\xi = \Delta^{\widetilde{L}}_{\eta_1 \eta_2 \ldots \eta_k \ldots}.$$

1 Geometry of the $\widetilde{L}$-representation of a real number $x \in (0, 1]$

Definition 1.1. Let $(c_1, c_2, \ldots, c_n)$ be a given tuple of positive integers. The cylinder of n-th rank with the base $c_1, c_2, \ldots, c_n$ is the set

$$\Delta^{\widetilde{L}}_{c_1 c_2 \ldots c_n} = \left\{x : x = \widetilde{L}(c_1, c_2, \ldots, c_n, a_{n+1}, a_{n+2} \ldots),\ a_{n+i} \in \mathbb{N},\ \forall i \in \mathbb{N}\right\}.$$

Cylinders have the following properties [6]:

(1) $\Delta^{\widetilde{L}}_{c_1 \ldots c_n} = \bigcup_{i=1}^{\infty} \Delta^{\widetilde{L}}_{c_1 \ldots c_n i}$.

(2) $\sup \Delta^{\widetilde{L}}_{c_1 \ldots c_{2m-1} i} = \inf \Delta^{\widetilde{L}}_{c_1 \ldots c_{2m-1}(i+1)}$, $\inf \Delta^{\widetilde{L}}_{c_1 \ldots c_{2m} i} = \sup \Delta^{\widetilde{L}}_{c_1 \ldots c_{2m}(i+1)}$.

(3) The cylinder $\Delta^{\widetilde{L}}_{c_1 \ldots c_n}$ is a half-open interval $(l_1, l_2]$ if n is odd or half-closed interval $[l_2, l_1)$ if n is even, where $l_1 = \widetilde{L}(c_1, \ldots, c_n + 1)$, $l_2 = \widetilde{L}(c_1, \ldots, c_n)$.

(4) $\operatorname{diam} \Delta^{\widetilde{L}}_{c_1 \ldots c_n} \equiv |\Delta^{\widetilde{L}}_{c_1 \ldots c_n}| = \frac{1}{c_1(c_1+1) \ldots c_n(c_n+1)} \leq \frac{1}{2^n} \xrightarrow{n \to \infty} 0$.

(5) If $d_j(a) = d_j(b)$ for $j < m$ and $d_m(a) > d_m(b)$, then $a < b$ for $m = 2n - 1$ and $a > b$ for $m = 2n$.

(6) Any permutation of $\widetilde{L}$-symbols in the base of cylinder does not change its length.

(7) The following equivalence holds:

$$\frac{|\Delta^{\widetilde{L}}_{c_1 \ldots c_m i}|}{|\Delta^{\widetilde{L}}_{c_1 \ldots c_m}|} = \frac{1}{i(i+1)} \iff |\Delta^{\widetilde{L}}_{c_1 \ldots c_m i}| = \frac{1}{i(i+1)} |\Delta^{\widetilde{L}}_{c_1 \ldots c_m}|.$$

(8) We have

$$|\Delta^{\widetilde{L}}_{c_1\dots c_m a}| = \sum_{j=a(a+1)}^{\infty} |\Delta^{\widetilde{L}}_{c_1\dots c_m j}|.$$

(9) It holds

$$\frac{|\Delta^{\widetilde{L}}_{c_1\dots c_m (i+1)}|}{|\Delta^{\widetilde{L}}_{c_1\dots c_m i 1}|} = \frac{2i}{i+2}.$$

Let (V_n) be a sequence of subsets of $\mathbb{N}$. The set $C[\widetilde{L}, (V_n)]$ is defined by the equality

$$C[\widetilde{L}, (V_n)] = \left\{x : x = \Delta^{\widetilde{L}}_{a_1 a_2 \dots a_n \dots},\ a_n(x) \in V_k \subset \mathbb{N},\ n = 1, 2, \dots\right\}.$$

Theorem 1.2. *The set $C[\widetilde{L}, (V_n)]$ is:*

(1) *a half-interval* $(0, 1]$ *to within a calculating set if all $V_n = \mathbb{N}$, $n \in \mathbb{N}$,*

(2) *the union of cylinders of rank m if $V_j = \mathbb{N}$ for $j > m$,*

(3) *a nowhere dense set if $V_n \neq \mathbb{N}$ for infinitely many n; moreover, the Lebesgue measure of the set is defined by the equality*

$$\lambda(C[\widetilde{L}, (V_n)]) = \prod_{n=1}^{\infty} \left[1 - \frac{\lambda(\overline{F}_n)}{\lambda(F_{n-1})}\right],$$

where

$$F_n = \bigcup_{a_1 \in V_1} \bigcup_{a_2 \in V_2} \cdots \bigcup_{a_n \in V_n} \Delta^{\widetilde{L}}_{a_1 \dots a_n}, \quad \overline{F}_n = F_{n-1} \setminus F_n.$$

Proof. Statements (1) and (2) are evident due to the previous theorem and the equality

$$C[\widetilde{L}, (V_n)] = \bigcup_{i_1 \in V_1} \cdots \bigcup_{i_m \in V_m} \Delta^{\widetilde{L}}_{i_1 \dots i_m}.$$

(3) For any interval $(a, b) \subset (0, 1]$, it is easy to find a cylinder $\Delta^{\widetilde{L}}_{c_1 \dots c_k} \subset (a, b)$. Then the interval

$$(\alpha, \beta) \equiv \operatorname{int} \Delta^{\widetilde{L}}_{c_1 \dots c_k \dots c_{n-1} j}, \quad \text{where } V_n \neq \mathbb{N} \text{ and } j \in \mathbb{N} \setminus V_n,$$

does not contain any point of the set $C[\widetilde{L}, (V_n)]$. So, this set is nowhere dense by definition.

The following equality follows from the definition of sets $C[\widetilde{L},(V_n)]$, F_n and $\overline{F}_n$ and the continuity of the Lebesgue measure λ:

$$\begin{aligned}\lambda(C[\widetilde{L},\{V_n\}]) &= \lim_{n\to\infty}\lambda(F_n)\\ &= \lim_{n\to\infty}\frac{\lambda(F_n)}{\lambda(F_{n-1})}\cdot\frac{\lambda(F_{n-1})}{\lambda(F\lambda_{n-2})}\cdots\frac{\lambda(F_2)}{\lambda(F_1)}\cdot\frac{\lambda(F_1)}{\lambda(F_0)}\\ &= \lim_{n\to\infty}\prod_{i=1}^{n}\frac{\lambda(F_i)}{\lambda(F_{i-1})}\\ &= \prod_{n=1}^{\infty}\frac{\lambda(F_n)}{\lambda(F_{n-1})} = \prod_{n=1}^{\infty}\left[1-\frac{\lambda(\overline{F}_n)}{\lambda(F_{n-1})}\right].\end{aligned}$$ □

Corollary 1.3. *The set*

$$C[\widetilde{L},V] = \left\{x : x = \Delta^{\widetilde{L}}_{a_1a_2\ldots a_n\ldots},\ a_n(x)\in V\subset\mathbb{N}\right\}$$

is:

(1) *a half-interval* $(0,1]$ *to within a calculating set, when* $V=\mathbb{N}$*,*

(2) *a nowhere dense nonclosed set of zero Lebesgue measure coinciding with its closure with respect to countable set when* $V\neq\mathbb{N}$*,*

(3) *self-similar if* V *is a finite set and* $\mathbb{N}$*-self-similar if* V *is an infinite set; moreover, its self-similar (*$\mathbb{N}$*-self-similar) dimension* α_s *is a solution of the equation*

$$\sum_{v\in V}\left(\frac{1}{v(v+1)}\right)^x = 1 \quad \textit{if } |V|<\infty, \tag{1.1}$$

and

$$\alpha_s = \sup_n\left\{x : \sum_{v:V\ni v\le n}\left(\frac{1}{v(v+1)}\right)^x = 1\right\} \quad \textit{if } |V|=\infty. \tag{1.2}$$

2 Structure and properties of the probability distribution function of the random variable with independent elements of the alternating Lüroth series

We consider the random variable $\xi = \Delta^{\widetilde{L}}_{\eta_1\eta_2\ldots\eta_k\ldots}$, where (η_k) is a sequence of independent random variables taking the values $1,2,\ldots,i,\ldots$ with probabilities $p_{1k},p_{2k},\ldots,p_{ik},\ldots$ respectively, $p_{ik}\ge 0$, $p_{1k}+p_{2k}+\cdots=1$ for all $k\in\mathbb{N}$. The numbers p_{mk} completely determine the distribution of the random variable ξ.

Theorem 2.1. *If the random variable* ξ *has a uniform distribution on* $[0, 1]$, *then the* $\widetilde{L}$*-symbols* η_k $(k = 1, 2, \dots)$ *are independent and identically distributed; moreover,*

$$P\{\eta_k = i\} = \frac{1}{i(i+1)}, \quad i = 1, 2, \dots.$$

Proof. Since ξ has a uniform distribution on [0, 1], we have

(1) $P\{\xi = a\} = 0$ for any $a \in [0, 1]$,

(2) $P\{\xi \in (a, b)\} = P\{\xi \in [a, b]\} = P([a, b]) = b - a.$

From property (4) of the cylinders $\Delta^{\widetilde{L}}_{c_1\dots c_m}$ it follows that

$$P(\Delta^{\widetilde{L}}_{c_1\dots c_m}) = |\Delta^{\widetilde{L}}_{c_1\dots c_m}| = \prod_{i=1}^{m} \frac{1}{c_i(c_i+1)}.$$

Since the distribution of the random variable η is continuous, we get

$$P\{\eta_1 = i\} = P\{\xi \in \Delta^{\widetilde{L}}_{i}\} = P(\Delta^{\widetilde{L}}_{i}) = |\Delta^{\widetilde{L}}_{i}| = \frac{1}{i(i+1)}$$

and

$$\begin{aligned} P\{\eta_2 = i\} &= P\left\{\xi \in \bigcup_{j=1}^{\infty} \Delta^{\widetilde{L}}_{ji}\right\} = P\left(\bigcup_{j=1}^{\infty} \Delta^{\widetilde{L}}_{ji}\right) \\ &= \sum_{j=1}^{\infty} |\Delta^{\widetilde{L}}_{ji}| = \sum_{j=1}^{\infty} \frac{1}{j(j+1)i(i+1)} \\ &= \frac{1}{i(i+1)} \sum_{j=1}^{\infty} \frac{1}{j(j+1)} = \frac{1}{i(i+1)}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} P\{\eta_{k+1} = i\} &= P\left\{\xi \in \bigcup_{j_1=1}^{\infty} \cdots \bigcup_{j_k=1}^{\infty} \Delta^{\widetilde{L}}_{j_1\dots j_k i}\right\} = \sum_{j_1=1}^{\infty} \cdots \sum_{j_k=1}^{\infty} |\Delta^{\widetilde{L}}_{j_1\dots j_k i}| \\ &= \frac{1}{i(i+1)} \sum_{j_1=1}^{\infty} \cdots \sum_{j_k=1}^{\infty} \frac{1}{j_1(j_1+1)\dots j_k(j_k+1)} = \frac{1}{i(i+1)}. \end{aligned}$$

Since the last probability does not depend on k and depends only on i, η_k are identically distributed.

Let us prove that for any $k, l \in \mathbb{N}$, $k < l$, random variable η_k does not depend on the random variable η_l and the following equality holds:

$$P\{\eta_1 = i, \eta_2 = j\} = P\{\eta_1 = i\} \cdot P\{\eta_2 = j\}$$

and

$$\begin{aligned}
&P\{\eta_k = i, \eta_l = j\} \\
&\quad = P\left\{\xi \in \bigcup_{j_1=1}^{\infty} \cdots \bigcup_{j_{k-1}=1}^{\infty} \bigcup_{j_{k+1}=1}^{\infty} \cdots \bigcup_{j_{l-1}=1}^{\infty} \Delta^{\widetilde{L}}_{j_1 \ldots j_{k-1} i j_{k+1} \ldots j_{l-1} j}\right\} \\
&\quad = \sum_{j_1=1}^{\infty} \cdots \sum_{j_{k-1}=1}^{\infty} \sum_{j_{k+1}=1}^{\infty} \cdots \sum_{j_{l-1}=1}^{\infty} |\Delta^{\widetilde{L}}_{j_1 \ldots j_{k-1} i j_{k+1} \ldots j_{l-1} j}| \\
&\quad = \frac{1}{i(i+1)j(j+1)} \sum_{j_1=1}^{\infty} \cdots \sum_{j_{k-1}=1}^{\infty} \sum_{j_{k+1}=1}^{\infty} \cdots \sum_{j_{l-1}=1}^{\infty} \frac{1}{j_1(j_1+1) \ldots j_{k-1}} \\
&\qquad\qquad \times \frac{1}{(j_{k-1}+1) j_{k+1}(j_{k+1}+1) \ldots j_{l-1}(j_{l-1}+1)} \\
&\quad = \frac{1}{i(i+1)} \cdot \frac{1}{j(j+1)} = P\{\eta_k = i\} \cdot P\{\eta_l = j\}. \qquad \square
\end{aligned}$$

Theorem 2.2. *The random variable ξ with independent $\widetilde{L}$-symbols has a discrete distribution if and only if*

$$\prod_{k=1}^{\infty} \max_m p_{mk} > 0. \tag{2.1}$$

If the distribution is discrete, then the set of atoms of the distribution of the random variable ξ consists of a point x_0 such that $p_{a_k(x_0)k} = \max_m\{p_{mk}\}$ for any $k \in \mathbb{N}$, and for all points $x' \in (0,1)$ one has $p_{a_k(x')k} > 0$ and there exists an $m \in \mathbb{N}$ such that $a_j(x') = a_j(x_0)$ for $j \geq m$.

Proof. The number x is an atom of distribution of the random variable ξ if

$$\prod_{k=1}^{\infty} p_{a_k(x)k} > 0.$$

Necessity. Let the random variable ξ have a discrete distribution and let x be an atom of distribution. Suppose that the infinite product (2.1) diverges to 0. Then

$$P\{\xi = x\} = \prod_{k=1}^{\infty} p_{a_k(x)k} \leq \prod_{k=1}^{\infty} \max_m p_{mk} = 0,$$

but this contradicts the fact that x is an atom of distribution. Therefore, this contradiction proves the necessity.

Sufficiency. Let the statement (2.1) hold. Let x' differ from x_0 for a finite number of $\widetilde{L}$-symbols such that $p_{a_k(x')k} > 0$. Then x_0 and all such x' are atoms of the distribution of ξ. Let us prove that the random variable ξ has a discrete distribution.

Let D_m be a set of all points x' such that $a_j(x') = a_j(x_0)$ for $j \geq m$. Then

$$P\{\xi \in D_m\} = \sum_{a_1(x')} \cdots \sum_{a_{m-1}(x')} \left(\prod_{k=1}^{m-1} p_{a_k(x')k} \cdot \prod_{k=m}^{\infty} p_{a_k(x_0)k} \right)$$
$$= \prod_{k=1}^{m-1} \sum_{a_k(x')} p_{a_k(x')k} \cdot \prod_{k=m}^{\infty} p_{a_k(x_0)k} = \prod_{k=m}^{\infty} p_{a_k(x_0)k}.$$

The set $D = \bigcup_{m=1}^{\infty} D_m$ is at most countable because it is a countable union of at most countable sets. Since

$$\{x_0\} = D_1 \subset D_2 \subset \cdots \subset D_m \subset D_{m+1} \subset \cdots,$$

by the continuity of probability

$$P\{\xi \in D\} = \lim_{m\to\infty} P\{\xi \in D_m\} = \lim_{m\to\infty} \prod_{k=m}^{\infty} p_{a_k(x_0)k} = 1.$$

From the properties of the convergent infinite products it follows that the last limit is equal to 1.

So the countable set D is the support of the distribution of the random variable ξ, that is, the distribution of ξ is discrete. □

Corollary 2.3. *The random variable ξ has a continuous distribution if and only if the infinite product* (2.1) *is equal to* 0.

Theorem 2.4. *A continuous random variable ξ with independent $\widetilde{L}$-symbols has either a pure absolutely continuous or a pure singularly continuous distribution.*

Proof. Let $\delta = (\delta_1 \ldots \delta_n)$ be an ordered tuple of positive integers and let T_δ^n be a *transformation* of a point $x = \widetilde{L}(a_1, \ldots, a_k, \ldots)$ such that

$$T_\delta^n(x) \equiv \widetilde{L}(\delta_1 \ldots \delta_n, a_1, \ldots, a_k, \ldots).$$

It is evident that the point $x_0 = \Delta^{\widetilde{L}}_{(\delta_1 \ldots \delta_n)}$ having a pure periodic $\widetilde{L}$-expansion with period $(\delta_1 \ldots \delta_n)$ is an invariant point of T_δ^n-transformation.

The T_δ^n*-transformation of the set* E is the set of T_δ^n-images of all $x \in E$, i.e.,

$$T_\delta^n(E) = \{u : u = T_\delta^n(x), \text{ where } x \in E\}.$$

It is easy to see that

$$T_\delta^n(0;1) = \Delta^L_{\delta_1\ldots\delta_n}$$

and T_δ^n-transformation is the similarity transformation with coefficient

$$k = \prod_{i=1}^n \frac{1}{\delta_i(\delta_i+1)}.$$

It is evident that $\lambda[T_\delta^n(E)] = k\lambda(E)$, where λ is Lebesgue measure. Therefore $\lambda[T_\delta^n(E)]$ and $\lambda(E)$ are equal to zero simultaneously. Let

$$T^n(E) = \bigcup_{\delta_1,\ldots,\delta_n} T_\delta^n(E), \quad T(E) = \bigcup_n T(E).$$

We consider the event $A = \{\xi \in T(E)\}$. The event A is generated by the sequence of independent random variables η_k and does not depend on all σ-algebras B_k generated by $\eta_1, \ldots, \eta_k$. So, A is residual. Therefore, from the Kolmogorov's law of 0 and 1 [5] it follows that $P(A) = 0$ or $P(A) = 1$.

Only one of the two cases is possible:

(1) there exists a set E such that $\lambda(E) = 0$ and $P\{\xi \in E\} > 0$;

(2) for an arbitrary set E such that $\lambda(E) = 0$ we have $P\{\xi \in E\} = 0$.

In the first case the equality $\lambda(E) = 0$ implies that $\lambda(T(E)) = 0$. Therefore, there is a set $T(E)$ such that $\lambda(T(E)) = 0$ and $P\{\xi \in T(E)\} = 1$. So, ξ has a pure singularly continuous distribution by definition. In the second case the distribution is pure absolutely continuous by definition. So, the distribution of the random variable ξ is pure. □

Theorem 2.5. *The continuous distribution of the random variable ξ is pure absolutely continuous if and only if*

$$\prod_{k=1}^{\infty}\left(\sum_{m=1}^{\infty}\sqrt{\frac{p_{mk}}{m(m+1)}}\right) > 0. \tag{2.2}$$

Proof. Let $\{(\Omega_k, B_k, \mu_k)\}$ and $\{(\Omega_k, B_k, \nu_k)\}$ be sequences of probability spaces such that

- $\Omega_k = \mathbb{N}$, B_k is a σ-algebra of all subsets of Ω_k,
- $\mu_k(m) = p_{mk}$, $\nu_k(m) = \frac{1}{m(m+1)}$, $k \in \mathbb{N}$,

where p_{mk} is an element of the matrix $\|p_{ik}\|$ that determines the distribution of the random variable ξ.

It is evident that $\mu_k \ll \nu_k$ for all $k \in \mathbb{N}$. Let us consider the infinite products of probability spaces

$$(\Omega, B, \mu) = \prod_{k=1}^{\infty} (\Omega_k, B_k, \mu_k), \quad (\Omega, B, \nu) = \prod_{k=1}^{\infty} (\Omega_k, B_k, \nu_k).$$

By using Kakutani's theorem [1], we have $\mu \ll \nu$ if and only if

$$\prod_{k=1}^{\infty} \rho(\mu_k, \nu_k) > 0,$$

where

$$\rho(\mu_k, \nu_k) = \int_{\Omega_k} \sqrt{\frac{d\mu_k}{d\nu_k}} d\nu_k$$

is the Hellinger integral. In this case

$$\prod_{k=1}^{\infty} \int_{\Omega_k} \sqrt{\frac{d\mu_k}{d\nu_k}} d\nu_k > 0 \iff \prod_{k=1}^{\infty} \left(\sum_{m=1}^{\infty} \sqrt{\frac{p_{mk}}{m(m+1)}} \right) > 0.$$

Therefore, from the condition (2.2) it follows that the measure μ is absolutely continuous with respect to the measure ν.

Let us consider the mapping $f : \Omega \to [0; 1]$ such that

$$f(\omega) = \Delta^{\widetilde{L}}_{\omega_1 \ldots \omega_k \ldots} \quad \text{for all } \omega = (\omega_1, \ldots, \omega_k, \ldots) \in \Omega.$$

For any Borel set E, we define the measures μ^* and ν^* as the image measures of μ and ν under f:

$$\mu^*(E) = \mu(f^{-1}(E)), \quad \nu^*(E) = \nu(f^{-1}(E)).$$

The measure μ^* coincides with the probabilistic measure P_ξ and the measure ν^* coincides with the probabilistic measure P_ψ, which is equivalent to the Lebesgue measure λ. From the absolutely continuity of the measure μ with respect to the measure ν it follows that the measure μ^* is absolutely continuous with respect to the measure ν^*. Since $\nu^* \sim \lambda$, from condition (2.2) it follows that the random variable ξ is of pure absolutely continuous distribution. □

Corollary 2.6. *The continuous distribution of the random variable ξ is pure singularly continuous if and only if*

$$\prod_{k=1}^{\infty} \left(\sum_{m=1}^{\infty} \sqrt{\frac{p_{mk}}{m(m+1)}} \right) = 0.$$

Lemma 2.7. *At a point $x = \Delta^{\widetilde{L}}_{a_1 a_2 \ldots a_k \ldots}$, the probability distribution function F_ξ of the random variable ξ is of the following form:*

$$F_\xi(x) = \beta_{a_1(x)1} + \sum_{k=2}^{\infty}\left(\beta_{a_k(x)k}\prod_{j=1}^{k-1} p_{a_j(x)j}\right), \tag{2.3}$$

where

$$\beta_{a_k(x)k} = \begin{cases} \sum_{j=a_k(x)+1}^{\infty} p_{jk}, & \text{if } k = 2m-1, \\ \sum_{j=1}^{a_k(x)-1} p_{jk}, & \text{if } k = 2m,\ m \in \mathbb{N}. \end{cases}$$

Proof. By the definition of the probability distribution function of random variable, $F_\xi(x) = P\{\xi < x\}$.

Since for the point $x = \widetilde{L}(a_1(x), a_2(x), \ldots, a_k(x), \ldots)$ the event $\{\xi < x\}$ is a union of exclusive events

$$\begin{aligned} \{\xi < x\} = \{\eta_1 > a_1(x)\} &\cup \{\eta_1 = a_1(x), \eta_2 < a_2(x)\} \\ &\cup \cdots \cup \{\eta_1 = a_1(x), \ldots, \eta_{2k-2} = a_{2k-2}(x), \eta_{2k-1} > a_{2k-1}(x)\} \\ &\cup \{\eta_1 = a_1(x), \ldots, \eta_{2k-1} = a_{2k-1}(x), \eta_{2k} < a_{2k}(x)\} \cup \cdots, \end{aligned}$$

we have

$$\begin{aligned} F_\xi(x) = \sum_{j=a_1(x)+1}^{\infty} p_{j1} &+ \sum_{j=1}^{a_2(x)-1} p_{j2}\cdot p_{a_1(x)1} \\ &+ \cdots + \sum_{j=a_{2k-1}(x)+1}^{\infty} p_{j,2k-1}\cdot\prod_{j=1}^{2k-2} p_{a_j(x)j} \\ &+ \sum_{j=1}^{a_{2k}(x)-1} p_{j,2k}\cdot\prod_{j=1}^{2k-1} p_{a_j(x)j} + \cdots \\ = \beta_{a_1(x)1} &+ \sum_{k=2}^{\infty}\left(\beta_{a_k(x)k}\prod_{j=1}^{k-1} p_{a_j(x)j}\right). \end{aligned}$$ □

Corollary 2.8. *The change δ in the probability distribution function F_ξ on the cylinder $\Delta^{\widetilde{L}}_{c_1 c_2 \ldots c_m}$ is calculated by the formula*

$$\delta \equiv \delta(\Delta^{\widetilde{L}}_{c_1 c_2 \ldots c_m}) = \prod_{i=1}^{m} p_{c_i i}.$$

Corollary 2.9. *If $p_{ck} = 0$, then the distribution function F_ξ is constant on each cylinder $\Delta^{\widetilde{L}}_{c_1c_2\ldots c_{k-1}c}$.*

Lemma 2.10. *If the probability distribution function F_ξ has a derivative (finite or infinite) at a point $x_0 = \Delta^{\widetilde{L}}_{a_1a_2\ldots a_n\ldots}$, then*

$$F'_\xi(x_0) = \prod_{i=1}^{\infty}(a_i(a_i + 1)p_{a_i i}).$$

Proof. In fact, if $F'_\xi(x_0)$ exists, then

$$F'_\xi(x_0) = \lim_{\substack{x'<x_0<x'' \\ x''-x'\to 0}} \frac{F_\xi(x'') - F_\xi(x')}{x'' - x'} = \lim_{n\to\infty} \frac{\delta(\Delta^{\widetilde{L}}_{a_1\ldots a_m})}{|\Delta^{L}_{a_1\ldots a_m}|}$$
$$= \lim_{m\to\infty} \prod_{i=1}^{m}(a_i(a_i + 1)p_{a_i i}). \qquad \square$$

3 Topological and metric properties of a singular distribution of the random variable ξ

Let us recall [5] that there are three types of singular probability distributions according to topological and metric properties of their spectra. The singular probability distribution of the random variable is called:

(1) the distribution of *Cantor type* (or *C-type*) if its spectrum S_ξ is a set of zero Lebesgue measure,

(2) the distribution of *Salem type* (or *S-type*) if its spectrum S_ξ contains closed intervals,

(3) the distribution of *quasi-Cantor type* (or *P-type*) if its spectrum S_ξ is a nowhere dense set of positive Lebesgue measure.

By definition, the spectrum of the distribution of the random variable is a minimal closed support of the distribution. Also the spectrum is a set of growth points of the probability distribution function.

Lemma 3.1. *The spectrum S_ξ of the distribution of the random variable ξ is the closure of the set*

$$B_\xi = \left\{x : x = \Delta^{\widetilde{L}}_{a_1a_2\ldots a_n\ldots},\ \text{where } p_{a_n(x)n} > 0 \text{ for all } n \in \mathbb{N}\right\} = C[\widetilde{L}, (V_n)].$$

Proof. Generally speaking, the set B_ξ is not closed. So, to prove the lemma it is enough to show that $B_\xi \subset S_\xi$ and any internal point of the set $[0,\ 1]\backslash B_\xi$ does not belong to S_ξ.

Let x' be a point such that $p_{a_j(x')j} > 0$ for any $j \in \mathbb{N}$. Let us show that x' belongs to the spectrum S_ξ.

By definition, the point x' is a point of growth of the probability distribution function F_ξ if for any $\varepsilon > 0$ the following inequality holds:

$$F_\xi(x' + \varepsilon) - F_\xi(x' - \varepsilon) > 0.$$

For any $\varepsilon > 0$, there exists a cylinder $\Delta^{\widetilde{L}}_{c_1c_2\dots c_m}$ such that

$$\Delta^{\widetilde{L}}_{c_1c_2\dots c_m} \subset (x' - \varepsilon,\ x' + \varepsilon)$$

and $x' \in \Delta^{\widetilde{L}}_{c_1c_2\dots c_m}$. Then

$$F_\xi(x' + \varepsilon) - F_\xi(x' - \varepsilon) \equiv \delta\big((x' - \varepsilon,\ x' + \varepsilon)\big) \geq \delta(\Delta^{\widetilde{L}}_{c_1c_2\dots c_m}) = \prod_{i=1}^{m} p_{c_i i} > 0.$$

If the point $x' \in [0,1]\backslash B_\xi$ is not the endpoint of any cylinder and there exists $p_{a_j(x')j} = 0$, then from Corollary 2.9 of Lemma 2.7 it follows that x' belongs to the interval $\nabla_{c_1c_2\dots c_m} \equiv \operatorname{int} \Delta_{c_1c_2\dots c_m}$. For $\varepsilon > 0$ such that

$$(x' - \varepsilon, x' + \varepsilon) \subset \nabla_{c_1c_2\dots c_m}$$

we have

$$F_\xi(x' + \varepsilon) - F_\xi(x' - \varepsilon) \leq \delta(\Delta^{L}_{c_1c_2\dots c_m}) = 0.$$

Thus $x' \notin S_\xi$. □

Theorem 3.2. *The spectrum S_ξ of the distribution of the random variable ξ is:*

(1) *the closed interval* $[0,\ 1]$ *if the matrix $\|p_{ik}\|$ does not contain zeros,*

(2) *the union of closed intervals if the matrix $\|p_{ik}\|$ has zeros only in a finite number of columns,*

(3) *a nowhere dense set, and the Lebesgue measure is calculated by the formula*

$$\lambda(S_\xi) = \prod_{k=1}^{\infty} W_k, \quad \textit{where } W_k = \sum_{i:p_{ik}>0} \frac{1}{i(i+1)}, \quad k \in \mathbb{N}, \tag{3.1}$$

if the matrix $\|p_{ik}\|$ contains zeros in an infinite number of columns.

Proof. Statement (1) is obvious.

(2) Let $p_{ik} > 0$ for $i \in \mathbb{N}$, $k \geq m$. Since

$$P\{\xi \in \Delta^{\widetilde{L}}_{c_1\dots c_m i_{m+1}\dots i_{m+n}}\} = \Bigg(\prod_{c_i:p_{c_i i}>0} p_{c_i i}\Bigg) \cdot \prod_{j=m+1}^{m+n} p_{i_j j} > 0$$

for any tuple of positive integers $(i_{m+1}, \dots, i_{m+n})$, $n \in \mathbb{N}$, we have that F_ξ is strictly increasing on each cylinder $\Delta^{\widetilde{L}}_{c_1\ldots c_m}$ such that $p_{c_k k} > 0, k = 1, \ldots, m-1$. That is, S_ξ is the closure of the set

$$\bigcup_{c_i : p_{c_i i} > 0} \Delta^{\widetilde{L}}_{c_1\ldots c_m}.$$

(3) From the definition of the spectrum S_ξ and Lemma 2.7 it follows that for any $k \in \mathbb{N}$,

$$S_\xi \cap \Delta^{\widetilde{L}}_{a_1\ldots a_k} \begin{cases} = \varnothing & \text{if there exists an } m \le k \text{ such that } p_{a_m m} = 0, \\ \neq \varnothing & \text{if } \prod_{j=1}^{k} p_{a_j j} > 0. \end{cases}$$

Then

$$\begin{aligned} \lambda(S_\xi) &= 1 - \Bigg(\sum_{a_1 : p_{a_1 1} = 0} |\Delta^{\widetilde{L}}_{a_1}| + \sum_{\substack{a_1 : p_{a_1 1} > 0 \\ a_2 : p_{a_2 2} = 0}} |\Delta^{\widetilde{L}}_{a_1 a_2}| \\ &\qquad + \sum_{\substack{a_1, a_2 : p_{a_1 1} p_{a_2 2} > 0 \\ a_3 : p_{a_3 3} = 0}} |\Delta^{\widetilde{L}}_{a_1\ldots a_k}| + \cdots \Bigg) \\ &= 1 - M_1 - W_1 M_2 - W_1 W_2 M_3 - \cdots \\ &= W_1 - W_1 M_2 - W_1 W_2 M_3 - \cdots \\ &= W_1 W_2 - W_1 W_2 M_3 - \cdots = \prod_{k=1}^{\infty} W_k, \end{aligned}$$

where $M_k = \sum_{i : p_{ik} = 0} \frac{1}{i(i+1)} = 1 - W_k, k \in \mathbb{N}$. □

Corollary 3.3. *Spectrum of the distribution of the random variable ξ is the set of zero Lebesgue measure if and only if*

$$\sum_{k=1}^{\infty} M_k = \infty.$$

This corollary follows from the theorem on the relation between infinite products and infinite series.

Theorem 3.4. *The singular distribution of the random variable ξ is a singular distribution:*

(1) *of C-type if and only if the infinite product* (3.1) *diverges,*

(2) *of S-type if and only if the matrix $\|p_{ik}\|$ has a finite number of columns containing zeros,*

(3) *P-type if and only if the infinite product* (3.1) *converges and the matrix $\|p_{ik}\|$ has an infinite number of columns containing zeros.*

Proof. Theorem 3.4 is a consequence of Theorem 3.2. In fact, if the matrix $\|p_{ik}\|$ has only a finite number of columns containing zeros, then the spectrum of the distribution of ξ is the union of closed intervals and the distribution is of S-type. If the matrix $\|p_{ik}\|$ has an infinite number of such columns, then the spectrum is a nowhere dense set. Moreover, the spectrum is a set of zero Lebesgue measure if the infinite product in (3.1) diverges to zero. So, ξ has the distribution of C-type. The spectrum is a set of positive Lebesgue measure if the infinite product in (3.1) converges. So, ξ has the distribution of P-type. Since these conditions are incompatible, this proves the theorem. □

4 Distributions of random variables with identically distributed $\widetilde{L}$-symbols

Let $\xi_0 = \Delta^{\widetilde{L}}_{\eta_1\eta_2\ldots\eta_k\ldots}$ be a random variable such that the $\widetilde{L}$-symbols η_k are independent and identically distributed, i.e., $P\{\eta_k = i\} = p_{ik} = p_i$. Then in the formula of the probability distribution function we have

$$\beta_{a_k(x)k} \equiv \beta_{a_k(x)} = \begin{cases} \sum_{j=a_k(x)+1}^{\infty} p_j, & \text{if } k = 2m-1, \\ \sum_{j=1}^{a_k(x)-1} p_j, & \text{if } k = 2m,\ m \in \mathbb{N}. \end{cases}$$

From Theorems 2.1 and 2.2 follows that the distribution of the random variable ξ_0 is a degenerate distribution (discrete distribution with a single atom) if $\max_i p_i = 1$ or a uniform distribution if $p_i = \frac{1}{i(i+1)}$ for all $i \in \mathbb{N}$ or a singular continuous distribution in other cases.

Thus singularity dominates in the studied class of distributions of random variables.

Lemma 4.1. *The graph Γ of the probability distribution function is an $\mathbb{N}$-self-affine set of the space $\mathbb{R}^2$ and*

$$\Gamma = \bigcup_{i=1}^{\infty} \varphi_i(\Gamma), \tag{4.1}$$

where

$$\varphi_i : \begin{cases} x' = \frac{1-x}{i(i+1)} + \frac{1}{i+1}, \\ y' = p_i(1-y) + \beta_i, \end{cases} \qquad \varphi_i(\Gamma) \cap \varphi_{i+1}(\Gamma) = C_i\left(\frac{1}{i+1}; \beta_i\right). \tag{4.2}$$

Proof. To prove equality (4.1) we firstly show that

$$\varphi_1(\Gamma) \cup \varphi_2(\Gamma) \cup \cdots \cup \varphi_n(\Gamma) \cup \cdots \equiv G \subset \Gamma.$$

To this end we consider any point M of the graph Γ

$$M(x; F_{\xi_0}(x)) \xrightarrow{\varphi_i} M_i\left(\frac{1-x}{i(i+1)} + \frac{1}{i+1}; \beta_i + p_i(1 - F_{\xi_0}(x))\right) = \varphi_i(M) \in \Gamma.$$

Now we show that $\Gamma \subset G$. Let $M(x;\, F_{\xi_0}(x)) \in \Gamma$. We consider the number $x_1 = \Delta^{\widetilde{L}}_{a_2(x)a_3(x)\ldots}$. Since $d_1(x) \in \mathbb{N}$, we have

$$F_{\xi_0}(x) = \beta_i + p_i(1 - F_{\xi_0}(x)).$$

From $\overline{M}(x_1; F_{\xi_0}(x)) \in \Gamma$ it follows that

$$\varphi_i(\overline{M}) = M(x; F_{\xi_0}(x)) \in G.$$

Equality (4.1) is proved.

Since

$$O(0;0) \xrightarrow{\varphi_i} C_{i-1}\left(\frac{1}{i}; \beta_{i-1}\right), \quad C(1;1) \xrightarrow{\varphi_i} C_i\left(\frac{1}{i+1}; \beta_i\right), \quad i \in \mathbb{N},$$

we have equality (4.2). □

Theorem 4.2. *The definite Lebesgue integral of the probability distribution function F_{ξ_0} on the closed interval* $[0, 1]$ *is calculated by the formula*

$$\int_0^1 F_{\xi_0}(x)dx = \frac{\sum_{i=1}^{\infty} \frac{\beta_{i-1}}{i(i+1)}}{1 + \sum_{i=1}^{\infty} \frac{p_i}{i(i+1)}}. \tag{4.3}$$

Proof. Since the Lebesgue integral has the additive property, we have

$$\begin{aligned}
I &\equiv \int_0^1 F_{\xi_0}(x)dx \\
&= \sum_{i=1}^{\infty} \int_{\frac{1}{i+1}}^{\frac{1}{i}} [\beta_i + p_i(1 - F_{\xi_0}(\Delta^{\widetilde{L}}_{d_2 d_3 \ldots d_n \ldots}))]dx \\
&= \sum_{i=1}^{\infty} \frac{\beta_i}{i(i+1)} + \left(\sum_{i=1}^{\infty} \frac{p_i}{i(i+1)}\right) \cdot \left(1 - \int_0^1 F_{\xi_0}(x)dx\right) \\
&= \sum_{i=1}^{\infty} \frac{\beta_i}{i(i+1)} + \sum_{i=1}^{\infty} \frac{p_i}{i(i+1)} - \left(\sum_{i=1}^{\infty} \frac{p_i}{i(i+1)}\right) \cdot \int_0^1 F_{\xi_0}(x)dx.
\end{aligned}$$

Then

$$\left(1+\sum_{i=1}^{\infty}\frac{p_i}{i(i+1)}\right)\cdot I=\sum_{i=1}^{\infty}\frac{\beta_{i-1}}{i(i+1)}.$$

So, we have (4.3). □

Theorem 4.3. *The probability distribution function of the random variable* ξ_0 *with independent identically distributed* $\widetilde{L}$*-symbols preserves the Hausdorff–Besicovitch dimension if and only if* $p_i=\frac{1}{i(i+1)}$ *for any* $i\in\mathbb{N}$.

Proof. Necessity. If $p_i=\frac{1}{i(i+1)}$ for all $i\in\mathbb{N}$, then $F_\xi(x)=x$. The function $F_\xi(x)$ is an identical transformation on $(0,1]$. Then the probability distribution function preserves the Hausdorff–Besicovitch dimension.

Sufficiency. Suppose that the probability distribution function preserves the Hausdorff–Besicovitch dimension and there exists a $j\in\mathbb{N}$ such that

$$p_j\neq\frac{1}{j(j+1)}.$$

Without loss of generality we assume that

$$p_j<\frac{1}{j(j+1)}. \tag{4.4}$$

Then there exists $p_m>\frac{1}{m(m+1)}$.

In fact, if $p_i\leq\frac{1}{i(i+1)}$ for $i\neq j$, then

$$1-p_j=\sum_{i\neq j\in\mathbb{N}}p_i\leq\sum_{i\neq j\in\mathbb{N}}\frac{1}{i(i+1)}=1-\frac{1}{j(j+1)}.$$

This contradicts inequality (4.4).

Let us consider p_c where $j\neq c\neq m$. Then

$$p_c\leq\frac{1}{c(c+1)}\quad\text{or}\quad p_c\geq\frac{1}{c(c+1)}.$$

We choose two numbers among the numbers p_j, p_m, p_c such that the inequalities of the same sign hold. Let p_l and p_k be such numbers.

Let us consider the set $C[\widetilde{L},V]$ containing only numbers whose $\widetilde{L}$-symbols belongs to the set $V=\{l,k\}$. It is a self-similar fractal. The Hausdorff–Besicovitch dimension of $C[\widetilde{L},V]$ coincides with the self-similar dimension and is a solution of the equation

$$\left(\frac{1}{l(l+1)}\right)^x+\left(\frac{1}{k(k+1)}\right)^x=1.$$

The image of this set under the transformation F_ξ is also self-similar fractal such that the Hausdorff–Besicovitch dimension is a solution of the equation

$$p_l^x + p_k^x = 1.$$

But it is obvious that these numbers do not coincide. □

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Author information

Mykola Pratsiovytyi, Pirogova Str. 9, 01030 Kyiv, Ukraine.
E-mail: prats4@yandex.ru

Yuriy Khvorostina, Pirogova Str. 9, 01030 Kyiv, Ukraine.
E-mail: khvorostina13@mail.ru