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ALGORITHMS AND ELEMENTARY FUNCTIONS: TWO SIDES OF THE SAME FUNDAMENTAL NOTION

The problem formulation: An elementary function is one of the foundational notions of calculus course. However, many calculus textbooks do not provide students with a clear definition of the elementary function or simply avoid it completely. The current paper offers a simple and rigor approach of introducing the notion of an elementary function via linear algorithms.

Discussion of the problem

1. The notion of an algorithm

First, we recall that the concept of algorithm is intuitive. In general, we can describe it a step-by-step procedure for producing a solution to a given problem. A clear and logically completed algorithmic block scheme is a great visual support for instruction. There is a well-established tradition to use these schemes not only to illustrate solutions or proofs, but mainly to reflect graphically some sophisticated concepts (Triola, 2004, 272). Other good examples are proves of the theorems written in the traditional form in many Geometry textbooks (see, for example, Alexander and Koeberlein, 1999, 104). More specific information regarding algorithms in the secondary school mathematics can be found in the collection of works NCTM (1998).

There are dozens of different explanations (some time mistakenly called definitions in some broader, rough, and nonmathematical meaning) of the concept of algorithm. In Mingus (1998), the following, more suitable description of the general notion of algorithm is given:

An algorithm is a computational recipe for the systematic execution of a procedure designed to solve a specific problem that maintains the following characteristics:

1. Input data along with a finite set of instructions are given.
2. A computing agent reacts to the input and instructions and carries out the steps.

3. Intermediate results are stored and used.

4. The computation is carried out in a discrete, stepwise fashion.

5. The computing agent interprets the set of instructions in such a way that computation is carried out deterministically, without resort to random methods. (34)

In Malcev (1980), the notion of algorithm is defined using the following restrictions:

1. The Discreteness of an Algorithm: an algorithm is a process of the sequential constructing of a value in a discrete time such that the main initial system of values is given at the beginning; at each following moment the system of values is derived by the given rule (program) from the system of values presented at a preceding moment.

2. The Determination of an Algorithm: the system of values at a present (non-initial) moment is uniquely determined by the system of values at a preceding moment of time.

3. The Elementary Character of an Algorithm Stages: the rule (program), by which at every moment a system of values is derived from the system of values presented at a preceding moment must be simple and local.

4. The Directedness of an Algorithm: if the way of determining of some value from a preceding value cannot be realized, then in this case the clear direction defining the result of the algorithm needs to be given.

5. The Array Property of an Algorithm: the initial system of values might be chosen from a potentially infinite set.

It is well known that the algorithms are equivalent to the enumerative functions. By the famous Church's thesis (Maltsev, 1980), the class of the latter coincides with the class of recursive functions. Thus, the notions of an algorithm and a recursive function are equivalent. This idea is a very constructive power tool allowing us to justify the existence and, more importantly, inexistence of any possible algorithmic process. We need to admit that behind any mathematical concept there is an algorithm realizing computations or proofs.

2. Elementary functions

We will apply the concept of algorithm for clarification of some very basic ideas of Calculus course. All key concepts of Calculus deal with elementary functions, such as polynomials, rational functions, exponential functions, logarithmic functions, trigonometric and inverse trigonometric functions, and so on. Well-known examples of non-elementary functions are the Dirichlet function ($f(x) = 0$ for all rational numbers x; and $f(x) = 1$ for all irrational x); the function $f(x) = [x]$ (the whole part of a number), $f(x) = \int \sin(x^2)\, dx$, and others. We cannot expect clear understanding of central Calculus theorems such as theorems of continuity, differentiability, and integrability, without detailed comprehension of the notion of elementary functions. Actually, one can say that the traditional Calculus course in general is the mathematical analysis of elementary functions. At the same time, there is a strange paradox: some Calculus textbooks do not contain even the term "elementary functions" (Smith, 1996; Fincy, 2001; Barnett, 1996); other popular textbooks and dictionaries just substitute the definition of an elementary function with some approximate narrative descriptions, or just assume that this notion is intuitively known (Borowski, 1991; Thomas, 2000; Jonston, 2001). One also cannot find a reasonable consistency in the textbooks, in trying to get a more or les rigorous definition of the elementary functions. There is a relatively good definition of the elementary function extracted from the popular Calculus textbook Larson (1994):

An elementary function is one that can be formed as sum, product, or composition of functions from the following list: polynomial functions, rational functions, functions involving radicals, logarithmic functions, exponential functions, trigonometric functions, inverse trigonometric functions.

Notice that there is no keyword “finite” in this definition. Also, the list of basic elementary functions, from which one can obtain all others, is questionable.

The best definition of an elementary function we can find in the article Vilenkin (1978). According to this article, the class of elementary function is the set of functions containing all basic elementary functions and closed by the operations of addition, multiplication, and composition.

The list of the basic elementary functions includes only the following functions:

all constants;

the identity function f(x) = x;

the function f(x) = 1/x, x ≠ 0;

the exponential function f(x) = ex;

the logarithmic function f(x) = lnx, x>0;

the trigonometric function f(x) = sin x;

the inverse trigonometric function f(x) = arcsin x, -π/2 ≤ x ≤ π/2.

Using some group theoretical tools, the authors clearly explain why those functions were chosen, and proved that any elementary function can be built from them. Let us briefly explain this choice. The main ideas of the classification in Vilenkin (1978) is that the class of functions from the general Calculus could be described by the following: select some main functions from this class and define the operations to be applied to these functions in order to get a function from this class (the closure property). Based on this concept, the authors proved that the above mentioned set of basic elementary functions is sufficient for the Calculus needs. Another interesting and central question is: why do we choose the set of basic functions listed above? In Vilenkin (1978), the authors observe that the multiplicative group Rx of the field R of all real numbers is a direct sum of the group D of order two and the multiplicative group R+ of all positive real numbers. The group D defines the sign of a number and the group R+ its absolute value. One of the remarkable mathematical facts is that the groups R+ and Q (the additive group of the field R) are isomorphic. This way we justify the existence of the functions $y = e^x, y = \ln x (x > 0)$ from the list above. The justification for trigonometric functions is coming from the definition of the exponential function of a complex variable. It is interesting to note that this approach is also based on the homomorphism of the additive group of real numbers on the group $T = \{e^{iy} \mid y \in R\}$, or on the multiplicative group of all complex numbers with the module is 1. The function $y = \arcsin x$ is included in the list as the inverse function for $y = \sin x$.

Using the above definition, one can easily prove that all rational functions (including polynomials), all power functions with any real exponent, trigonometric and inverse trigonometric functions are elementary. The operations of addition and multiplication, and inverse operations of subtraction and division are chosen here as the main operations of a numerical field. The principle of selecting the above functions as the basic elementary functions can be rigorously justified using Group Theory methods. This approach definitely cannot be used for introducing elementary functions to beginners since it is based on some deep ideas of abstract algebra.

3. Elementary functions as linear algorithms

A numerical function is determined by its domain and its mapping relation, which is a computational algorithm allowing us calculate a value of the function corresponding to the given value of the independent variable. We can think of functions as computational algorithms. It is logical to select from the set of all functions the subset of elementary functions imposing some specific restrictions on these computational algorithms. From this standpoint, it is natural to give the following new definition.

A function is called an elementary function if it can be represented using a linear algorithm (an algorithm which does not include loops and branching), each node of which is computing a value of one of the basic elementary functions.

Note that our definition avoids the operations on functions including the composition of functions. Moreover, even more complicated for the beginner, the concept of a composite function becomes clear with the algorithmic approach. This could be useful in the study of such a critical topic as the rule of differentiation of composite functions (the chain rule). Another benefit of this approach is clarifying the concept of a mathematical formula as a short record of a computational algorithm. There are some simple examples of functions that are initially defined by branching algorithms but could be represented via linear algorithms. For instance, the function f(x) = | x | can be also defined as f (x) = $\sqrt{x^2}$. Note, that in the very interesting article Pugachov (1964), one find a general approach to representing of piece wise functions by one analytic formula. Let us briefly describe this approach.

Let us consider the following functions:

$$(1)\quad y=\begin{cases} f_1(x), -\infty < x < a_1; \\ f_2(x), a_1 < x < a_2; \\ \dots\dots\dots\dots\dots\dots \\ f_n(x), a_{n-1} < x < +\infty; \end{cases} \text{ and } (2)\quad y=\begin{cases} \varphi_1(x), -\infty < x \le a_1; \\ \varphi_2(x), a_1 < x \le a_2; \\ \dots\dots\dots\dots\dots\dots \\ \varphi_n(x), a_{n-1} < x < +\infty; \end{cases},$$

where $f_1(x), f_2(x), \dots, f_n(x)$, $\varphi_1(x)$, $\varphi_2(x), \dots$, $\varphi_n(x)$ are elementary functions, satisfying the conditions:

$$\varphi_1(a_1)=\varphi_2(a_1)=A_1, \varphi_2(a_2)=\varphi_3(a_2)=A_2, \dots, \varphi_{n-1}(a_{n-1})=\varphi_n(a_n)=A_{n-1}.$$

This construction does not allow us to find a value of *y* by using a linear algorithm. However, by using some auxiliary functions, it is possible to define (1) and (2) with a single formula. Let

$$u_1(x)=\frac{1}{2}\left(1-\frac{|x-a_1|}{x-a_1}\right),\ u_n(x)=\frac{1}{2}\left(1-\frac{|x-a_{n-1}|}{x-a_{n-1}}\right),$$

$$u_k(x)=\frac{1}{4}\left(1+\frac{|x-a_{k-1}|}{x-a_{k-1}}\right)\left(1+\frac{|x-a_k|}{x-a_k}\right), k=2,3,\dots,n-1$$

It is easy to see that (1) could be represented by the following single formula

$$y(x)=u_1(x)f_1(x)+u_2(x)f_2(x)+\dots+u_n(x)f_n(x).$$

Similarly for (2) we introduce the functions

$$v_1(x)=\frac{x+a_1-|x-a_1|}{2}, v_n(x)=\frac{x+a_{n-1}+|x-a_{n-1}|}{2},$$

$$v_k(x)=\frac{x+a_{k-1}+|x-a_{k-1}|}{2}-\frac{x-a_k+|x-a_k|}{2}, k=2,3,\dots,n-1.$$

Now the function (2) can be written as the following single expression

$$y(x)=\varphi_1(v_1(x))+\varphi_2(v_2(x))+\dots+\varphi_n(v_n(x))-\varphi_1(a_1)-\varphi_2(a_2)-\dots-\varphi_{n-1}(a_{n-1}).$$

So, since the functions (1) and (2) have been given by single formulas (i.e. by linear algorithms) on each interval from the given finite set of intervals, we were able to find a single linear algorithm defining each of these functions (1) and (2) on their domains.

Consider some examples.

We will use the following obvious statement *: If* $y(x)=\bar{y}(x)\forall x\in D(y)$ *where* $D(y)\subset D(\bar{y})$*, a* $D(y),D(\bar{y})-$ *domains of the functions* $y(x),\bar{y}(x)$ *and the function* $\bar{y}(x)-$ *is alementary, or continues, or differentiable on* $D(\bar{y})$*, then the function* $y(x)$*is also alementary, or continues, or differentiable on* $D(y)$.

Example. Prove that the following functions are elementary by representing them as functions written with one formula.

a)
$$y(x)=\begin{cases} x^3+x-1, -\infty<x<-5; \\ e^{2x} \quad , \quad -5<x<1; \\ 1000x \quad , \quad 1<x<20; \\ \sin\ln\left(x^2+1\right), \ 20<x<+\infty. \end{cases}$$

◄ This function $y(x)$ is a function of the type (1) above. It continues on its domain $D(y)=(-\infty;-5)\cup(-5;1)\cup(1;20)\cup(20;+\infty)$ and could be define by the following unique formula.

$$y(x)=\frac{1}{2}\left(1-\frac{|x+5|}{x+5}\right)\left(x^3+x-1\right)+\frac{1}{4}\left(1+\frac{|x+5|}{x+5}\right)\left(1-\frac{|x-1|}{x-1}\right)e^{2x}+$$
$$+\frac{1}{4}\left(1+\frac{|x-1|}{x-1}\right)\left(1-\frac{|x-20|}{x-20}\right)1000x+\frac{1}{2}\left(1+\frac{|x-25|}{x-20}\right)\sin\ln\left(x^2+1\right).$$

Indeed, if $-\infty<x<-5$, then

$$y(x)=1\cdot\left(x^3+x-1\right)+0\cdot e^{2x}+0\cdot 1000x+0\cdot\sin\ln\left(x^2+1\right)=x^3+x-1;$$

if $-5<x<1$, then

$$y(x)=0\cdot\left(x^3+x-1\right)+\frac{1}{4}(2)(2)\cdot e^{2x}+(0)(2)\cdot 1000x+0\cdot\sin\ln\left(x^2+1\right)=e^{2x};$$

if $1<x<20$, then

$$y(x)=0\cdot\left(x^3+x-1\right)+\frac{1}{4}(2)(0)\cdot e^{2x}+\frac{1}{4}(2)(2)\cdot 1000x+0\cdot\sin\ln\left(x^2+1\right)=1000x;$$

if $20<x<+\infty$,

then

$$y(x)=0\cdot\left(x^3+x-1\right)+\frac{1}{4}(2)(0)\cdot e^{2x}+\frac{1}{4}(2)(0)\cdot 1000x+1\cdot\sin\ln\left(x^2+1\right)=\sin\ln\left(x^2+1\right).$$

► ***Answer.***

$$y(x)=\frac{1}{2}\left(1-\frac{|x+5|}{x+5}\right)\left(x^3+x-1\right)+\frac{1}{4}\left(1+\frac{|x+5|}{x+5}\right)\left(1-\frac{|x-1|}{x-1}\right)e^{2x}+$$
$$+\frac{1}{4}\left(1+\frac{|x-1|}{x-1}\right)\left(1-\frac{|x-20|}{x-20}\right)1000x+\frac{1}{2}\left(1+\frac{|x-25|}{x-20}\right)\sin\ln\left(x^2+1\right).$$

б)
$$y(x)=\begin{cases} |x| \quad , -\infty<x\le 2; \\ 2\cos\pi x \quad , \quad 2<x<3; \\ \dfrac{4}{\pi}\arcsin\dfrac{x-6}{3}, \quad 3\le x\le 9; \\ \log_3 x \quad , 9<x\le 10. \end{cases}$$

◀ The function $y(x)$ is continues, and $y(x)=\bar{y}(x)$ on its domain $D(y)=(-\infty;10]$, where $D(y)=(-\infty;10]\subset D(\bar{y})=(-\infty;+\infty)$ i

$$\bar{y}(x)=\begin{cases} |x| & ,-\infty<x\le 2; \\ 2\cos\pi x & ,\quad 2\le x\le 3; \\ \dfrac{4}{\pi}\arcsin\dfrac{x-6}{3}, & 3\le x\le 9; \\ \log_3 x & ,\ 9\le x<+\infty. \end{cases}$$

If we show that $\bar{y}(x)$ is defined by one analytical expression (formula) on its domain $D(\bar{y})=(-\infty;+\infty)\supset(-\infty;10]=D(y)$, then the same formula will define the function $y(x)$ in every point of its domain $D(y)=(-\infty;10]\subset D(\bar{y})$. However $\bar{y}(x)$ is a function of type (2), that is why it defined by the formula

$$\bar{y}(x)=\left|\frac{x+2-|x-2|}{2}\right|+2\cos\pi\left(\frac{x+2+|x-2|}{2}-\frac{x-3+|x-3|}{2}\right)+$$

$$+\frac{4}{\pi}\arcsin\left(\frac{1}{3}\left(\frac{x+3+|x-3|}{2}-\frac{x-9+|x-9|}{2}\right)-2\right)+\log_3\left(\frac{x+9+|x-9|}{2}\right)-2+2-2=$$

$$=\left|\frac{x+2-|x-2|}{2}\right|+2\cos\pi\left(\frac{5+|x-2|-|x-3|}{2}\right)+$$

$$+\frac{4}{\pi}\arcsin\left(\left(\frac{12+|x-3|-|x-9|}{6}\right)-2\right)+\log_3\left(\frac{x+9+|x-9|}{2}\right)-2.$$

Indeed,

if $-\infty<x\le 2$, then

$$y(x)=\bar{y}(x)=\left|\frac{x+2+x-2}{2}\right|+2\cos\pi\left(\frac{5-x+2+x-3}{2}\right)+$$

$$+\frac{4}{\pi}\arcsin\left(\left(\frac{12-x+3+x-9}{6}\right)-2\right)+\log_3\left(\frac{x+9-x+9}{2}\right)-2=$$

$$=|x|+2\cos 2\pi+\frac{4}{\pi}\arcsin(-1)+\log_3 9-2=|x|+2+\frac{4}{\pi}\left(-\frac{\pi}{2}\right)+2-2=$$

$$=|x|+2-2+2-2=|x|;$$

if $2\le x\le 3$, then

$$y(x)=\bar{y}(x)=\left|\frac{x+2-x+2}{2}\right|+2\cos\pi\left(\frac{5+x-2+x-3}{2}\right)+$$

$$+\frac{4}{\pi}\arcsin\left(\left(\frac{12-x+3+x-9}{6}\right)-2\right)+\log_3\left(\frac{x+9-x+9}{2}\right)-2=$$

$$=2+2\cos\pi x-2+2-2=2\cos\pi x;$$

if $3\le x\le 9$, then

$$y(x)=\bar{y}(x)=\left|\frac{x+2-x+2}{2}\right|+2\cos\pi\left(\frac{5+x-2-x+3}{2}\right)+$$

$$+\frac{4}{\pi}\arcsin\left(\left(\frac{12+x-3+x-9}{6}\right)-2\right)+\log_3\left(\frac{x+9-x+9}{2}\right)-2=$$

$$=2+2\cos3\pi+\frac{4}{\pi}\arcsin\frac{x-6}{3}+\log_3 9-2=2-2+\frac{4}{\pi}\arcsin\frac{x-6}{3}+2-2=\frac{4}{\pi}\arcsin\frac{x-6}{3};$$

if $9\le x\le 10$, then

$$y(x)=\bar{y}(x)=\left|\frac{x+2-x+2}{2}\right|+2\cos\pi\left(\frac{5+x-2-x+3}{2}\right)+$$

$$+\frac{4}{\pi}\arcsin\left(\left(\frac{12+x-3-x+9}{6}\right)-2\right)+\log_3\left(\frac{x+9+x-9}{2}\right)-2=$$

$$=2+2\cos3\pi+\frac{4}{\pi}\arcsin1+\log_3 x-2=2-2+\frac{4}{\pi}\frac{\pi}{2}+\log_3 x-2=$$

$$=2-2+2+\log_3 x-2=\log_3 x.$$

►***Answer:***

$$y(x)=\left|\frac{x+2-|x-2|}{2}\right|+2\cos\pi\left(\frac{5+|x-2|-|x-3|}{2}\right)+$$

$$+\frac{4}{\pi}\arcsin\left(\left(\frac{12+|x-3|-|x-9|}{6}\right)-2\right)+\log_3\left(\frac{x+9+|x-9|}{2}\right)-2.$$

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Анотація. Субботін І., Білоцький Н. Алгоритми і елементарні функції: два боки одного поняття.

Поняття елементраної функції є одним з фундаментальних понять у курсі математичного аналізу. Водночас, у багатьох підручниках не пропонується чітке означення елементраної функції. У статті за допомогою поняття лінійного алгоритму надається простий, строгий і доступний студентам підхід до визначення поняття елементарної функції.

***Ключові слова**: алгоритми, елементарна функція.*

Аннотация. Субботин И., Билоцкий Н. Алгоритмы и элементарные функции: две стороны фундаментального понятия.

Понятие элементарная функции является одним из основополагающих в курсе матанализа. Тем не менее, многие учебники анализа не дают студентам четкого определения элементарной функции или просто избегают его полностью. В настоящем работе с помощью понятия линейного алгоритма предлагается простой, строгий и достунный студентам подход к понятию элементарной функции.

***Ключевые слова**: алгоритмы, элементарная функция.*

Abstract**. **Subbotin I., Bilotskii N. Algorithms and elementary functions: two sides of the same fundamental notion.

An elementary function is one of the foundational notions of calculus course. However, many calculus textbooks do not provide students with a clear definition of the elementary function or simply avoid it completely. The current paper offers a simple and rigor approach of introducing the notion of an elementary function via linear algorithms.

***Key words**: Algorithms, Elementary Function.*